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S_p-R_0 and S_p-R_1 Spaces

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Abstract— The purpose of this paper is to introduce two new classes of separation axioms by using the class of S_p-open sets called S_p-R_0 and S_p-R_1 topological spaces. Also investigate some of their characterization and fundamental properties of them and relationship of them with other separation axioms.

Index Terms — semi-open sets, S_p-open sets, S_p-closure, S_p-R_0 and S_p-R_1.

1 Introduction

The notion of S_p-open sets was introduced by Shareef H. A. in 2007 [M. Sc. Hardi] [1]. In 1963, N. Levine [7] offered a new notion to the field of general topology by introducing semi-open sets and defined this notion by utilizing the known notion of closure of an open set. Since the advent of this notion, several new notions are defined in terms of semi-open sets of which two are semi-R_0 and semi-R_1 introduced by S. N. Maheshwari and R. Prasad [3] and C. Dorsett [2], respectively. These two notions are defined as natural generalizations of the separation axioms R_0 which introduced by N. A. Shanin [6] in 1943 and R_1 by A. S. Davis [4] in 1961. In 1982 the notion of preopen sets was introduced by A. S. Mashhour et al. [5]. In this paper I study and introducing the mentioned separation axioms of topological spaces in terms of S_p-open sets called S_p-R_0 and S_p-R_1.

2 Preliminaries

Throughout this paper X and Y will denote topological spaces. If A is a subset of X, then the closure and interior of A in X are deonted by $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ respectively. The S_p-closure of A is denoted by $S_p\mathrm{cl}(A)$. While $\mathrm{scl}(A)$ denote the semi-closure of A. $S_pO(X)$ denote the family of all S_p-open sets in the space X.

Definition 2.1: A subset A of a space X is said to be:
1. Semi-open set [7], if $A \subseteq \mathrm{clint}(A)$.
2. Preopen set [5], if $A \subseteq \mathrm{intcl}(A)$.

The following definition and results are from [1].

Definition 2.2: A semi-open set A of a space X is called S_p-open set if for each $x \in A$, there exists a preclosed set F such that $x \in F \subseteq A$.

Definition 2.3: A subset N of a space X is said to be S_p-neighborhood (S_p-nbhd) of a point $x \in X$, if there exists an S_p-open set U such that $x \in U \subseteq N$.

Definition 2.4: Let X a space and $A \subseteq X$, then S_p-closure of A is the intersection of all S_p-closed sets which containing A.

Lemma 2.5: Let A and B be subsets of a space X, then:
1. $S_p\mathrm{cl}(A)$ is the smallest S_p-closed set containing A.
2. A is S_p-closed set if and only if $A = S_p\mathrm{cl}(A)$.
3. If $A \subseteq B$, then $S_p\mathrm{cl}(A) \subseteq S_p\mathrm{cl}(B)$.
4. $\mathrm{scl}(A) \subseteq S_p\mathrm{cl}(A)$.

Lemma 2.6: **[8]** For any subset A of a space X, $\mathrm{scl}(A) = \mathrm{intcl}(A) \cup A$.

The following are from [9].

Definition 2.7: A space X is said to be S_p-T_0 space if for each pair of distinct points in X, there exists an S_p-open set in X containing one of them and not the other.

Definition 2.8: A space X is said to be S_p-T_1 space if for each pair of distinct points x and y in X, there exists two S_p-open sets U and V in X containing x and y respectively such that $y \notin U$ and $x \notin V$.

Definition 2.9: A space X is said to be S_p-T_2 space if for each pair of distinct points x and y in X, there exists two disjoint S_p-open sets U and V in X such that x ∈ U and y ∈ V.

Theorem 2.10: A space X is S_p-T_1 if and only if every singleton subset of X is S_p-closed.

Definition 2.11: [3] A space X is said to be semi-R_0 if for each semi-open set U and $x \in U$, $\mathrm{scl}(\{x\}) \in U$.

Definition 2.12: [2] A space X is said to be semi-R_1 if for each $x, y \in X$ such that $\mathrm{scl}(\{x\}) \neq \mathrm{scl}(\{y\})$, then there exist disjoint semi-open sets U and V such that $\mathrm{scl}(\{x\}) \subseteq U$ and $\mathrm{scl}(\{y\}) \subseteq V$.

3 S_p-R_0 and S_p-R_1 Spaces

Definition 3.1: Let A be subset of a space X. The S_p-kernel of A, denoted by $S_p\text{-}\ker(A)$, is defined to be the set

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S_p-ker(A) = ∩ {$U \in S_pO(X)$: $A \subseteq U$}.

Definition 3.2: Let x be a point of a space X. The The S_p-kernel of x, denoted by S_p-ker ({x}), is defined to be the set S_p-ker({x}) = ∩ {$U \in S_pO(X)$: $x \in U$}.

Lemma 3.3: Let X be a space and A ⊆ X. Then S_p-ker(A) = {x ∈ X: $S_p$cl({x}) ∩ A ≠ ϕ}.
Proof: Let $x \in$ S_p-ker(A) and $S_p$cl({x}) ∩ $A = \phi$. Then $x \notin X \backslash S_p$cl({$x$}) which is an S_p-open set containing A this is a contradiction, so $S_p$cl({x}) ∩ $A \neq \phi$.
Now let $S_p$cl({x}) ∩ $A \neq \phi$ and $x \notin S_p$-ker(A). Then there exists an S_p-open set U containing A and $x \notin U$, and since $S_p$cl({x}) ∩ $A \neq \phi$ so let $y \in S_p$cl({x}) ∩ A. Since $A \subseteq U$, then U is S_p-nbhd of y and since $y \in S_p$cl({x}), then $U \cap \{x\} \neq \phi$ which is a contradiction. Therfore $x \in S_p$-ker(A), therefore S_p-ker (A) = {$x \in X$: $S_p$cl({x}) ∩ $A \neq \phi$}.

Lemma 3.4: Let X be a space and $x \in X$. Then $y \in S_p$-ker({x}) if and only if x ∈ $S_p$cl({y}).
Proof: Obvious.

Theorem 3.5: Let X be a space. Then the following statements are equivalent for any points x and y in X:

1. S_p-ker({x}) ≠ S_p-ker({y}).
2. $S_p$cl({x}) ≠ $S_p$cl({y}).

Proof: (1) ⇒ (2). Let S_p-ker({x}) ≠ S_p-ker({y}). Then there exists a point z in X such that $z \in S_p$-ker({x}) and $z \notin S_p$-ker({y}) this implies that by [Lemma 3.3] $S_p$cl({z}) ∩ {x} ≠ ϕ, then $x \in S_p$cl({z}) and since $z \notin S_p$-ker({y}) implies that $S_p$cl({z}) ∩ {y} = ϕ. But $x \in S_p$cl({z}), then $S_p$cl({x}) ⊆ $S_p$cl({z}) and since $S_p$cl({z}) ∩ {y} = ϕ implies that $S_p$cl({x}) ≠ $S_p$cl({y}).
(2) ⇒ (1). Let $S_p$cl({x}) ≠ $S_p$cl({y}). Then there exists a point $z \in S_p$cl({x}) and $z \notin S_p$cl({y}) implies that there exists an S_p-open set U in X containing z and x but not y, then $y \notin S_p$-ker({x}) and hence S_p-ker({x}) ≠ S_p-ker({y}).

Definition 3.6: A space X is said to be S_p-R_0 if for each S_p-open set U and each $x \in U$, implies $S_p$cl({x}) ⊆ U.

Definition 3.7: A space X is said to be S_p-R_1 if for x and y in X with $S_p$cl({x}) ≠ $S_p$cl({y}), there exists two disjoint S_p-open sets U and V such that $S_p$cl({x}) ⊆ U and $S_p$cl({y}) ⊆ V.

Lemma 3.8: If a space X is S_p-R_0 space, then for every S_p-open set U and each $x \in U$, intcl({x}) ⊆ U.
Proof: Let X be S_p-R_0 space. Then by [Definition 3.6] for every S_p-open set U and each $x \in U$, $S_p$cl({x}) ⊆ U. But by [Lemma 2.5] scl({x}) ⊆ $S_p$cl({x}) and scl({x}) = intcl({x}) ∪ {x} by [Lemma 2.6], this implies that intcl({x}) ⊆ U.

Lemma 3.9: Every S_p-R_1 space is S_p-R_0.
Proof: Let X be an S_p-R_1 space and U be any S_p-open set in X with $x \in U$. If $y \notin U$, then $x \notin S_p$cl({y}) implies that $S_p$cl({x}) ≠ $S_p$cl({y}) and since X is S_p-R_1 space so by there exists disjoint S_p-open sets G and H such that $S_p$cl({x}) ⊆ G and $S_p$cl({y}) ⊆ H. Now $x \notin H$ implies that $y \notin S_p$cl({x}). Thus $S_p$cl({x}) ⊆ U, hence X is S_p-R_0 space.

Remark 3.10: The converse of [Lemma 3.9] is not true ingeneral as $X = \{a,b,c,d\}$ and $\tau = \{\phi, X, \{c\}, \{b,d\}, \{b,c,d\}\}$. Then X is S_p-R_0 but not S_p-R_1 space since for $a, c \in X$ with $S_p$cl({a}) = {a} ≠ $S_p$cl({c}) = {c} there does not exist two disjoint S_p-open sets one of them contain $S_p$cl({a}) and the other contain $S_p$cl({c}).

Fromm [Definition 2.11] and [Definition 2.12] it is clear that every S_p-R_0 is semi-R_0 and every S_p-R_1 is semi-R_1 but the converse of them are not true ingeneral as shown in the following examples:

1. Let $X = \{a,b,c,d\}$ and $\tau = \{\phi, X, \{c\}, \{a,b\}, \{a,b,c\}\}$. Then $SO(X) = \{\phi, X, \{c\}, \{a,b\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}\}$ and $S_pO(X) = \{\phi, X, \{a,b\}, \{c,d\}, \{a,b,d\}\}$, now X is semi-R_0 but not S_p-R_0 since $\{a,b,d\} \in S_pO(X)$ while $S_p$cl({d}) = {c,d} is not a subset of {a,b,d}.
2. Let X be the same set as defined in (1) and $\tau = \{\phi, X, \{d\}, \{a,c\}, \{a,c,d\}\}$. Then X is semi-R_1 but not S_p-R_1.

Theorem 3.11: A space X is S_p-R_0 if and only if for every S_p-closed set F and $x \notin F$, there exists an S_p-open set G such that $x \notin G$ and $F \subseteq G$.
Proof: Let X be S_p-R_0 space and F be S_p-closed subset of X not containing $x \in X$. Then $X \backslash F$ is S_p-open set containing x, since X is S_p-R_0 space implies that $S_p$cl({x}) ⊆ $X \backslash F$ and then $F \subseteq X \backslash S_p$cl({$x$}). Now let $G = X \backslash S_p$cl({x}), then G is S_p-open set not contains x and $F \subseteq G$.
Conversely: Let $x \in G$ where G is S_p-open set in X. Then $X \backslash G$ is S_p-closed set and $x \notin X \backslash G$ implies that by hypothesis there exists and S_p-open set U such that $x \notin U$ and $X \backslash G \subseteq U$. Now $X \backslash U \subseteq G$ and $x \in X \backslash U$, but $X \backslash U$ is S_p-closed set then $S_p$cl({x}) ⊆ $X \backslash U \subseteq G$ this implies that X is S_p-R_0 space.

Theorem 3.12: For a space X, the following are equivalent:

1. X is S_p-R_0.
2. For any two points x and y in X, $x \in S_p$cl({y}) if and only if $y \in S_p$cl({x}).

Proof: (1) ⇒ (2). Let X be S_p-R_0 and $x \in S_p$cl({y}). To show $y \in$

$\mathrm{S_pcl}(\{x\})$, let V be any $\mathrm{S_p}$-open set containing y. Since X is $\mathrm{S_p}$-$\mathrm{R_0}$ space so $\mathrm{S_pcl}(\{y\}) \subseteq V$ implies that $x \in V$, hence every $\mathrm{S_p}$-open set which containing y contains x this implies that $y \in \mathrm{S_pcl}(\{x\})$. By the same way we can prove that if $y \in \mathrm{S_pcl}(\{x\}$, then $x \in \mathrm{S_pcl}(\{y\}$

(2) ⇒ (1). Let the hypothesis be satisfied and U be any $\mathrm{S_p}$-open set and $x \in U$. To show $\mathrm{S_pcl}(\{x\}) \subseteq U$, let $y \in \mathrm{S_pcl}(\{x\})$ implies that by hypothesis $x \in \mathrm{S_pcl}(\{y\})$, and then $U \cap \{y\} \neq \emptyset$ this implies that $y \in U$. Thus $\mathrm{S_pcl}(\{x\}) \subseteq U$, therefore X is $\mathrm{S_p}$-$\mathrm{R_0}$ space.

Theorem 3.13: A space X is $\mathrm{S_p}$-$\mathrm{R_0}$ if and only if for any x and y in X if $\mathrm{S_pcl}(\{x\}) \neq \mathrm{S_pcl}(\{y\})$, then $\mathrm{S_pcl}(\{x\}) \cap \mathrm{S_pcl}(\{y\}) = \phi$.

Proof: Let X be $\mathrm{S_p}$-$\mathrm{R_0}$ space and $x, y \in X$ such that $\mathrm{S_pcl}(\{x\}) \neq \mathrm{S_pcl}(\{y\})$. Then there exists $z \in \mathrm{S_pcl}(\{x\})$ such that $z \notin \mathrm{S_pcl}(\{y\})$ implies that there exists an $\mathrm{S_p}$-open set U containing z but not y, hence $x \in \mathrm{S_pcl}(\{x\})$. Therefore we have $x \notin \mathrm{S_pcl}(\{y\})$ implies that $x \in X \backslash \mathrm{S_pcl}(\{y\})$ which is an $\mathrm{S_p}$-open set, but X is $\mathrm{S_p}$-$\mathrm{R_0}$ so $\mathrm{S_pcl}(\{x\}) \subseteq X \backslash \mathrm{S_pcl}(\{y\})$ this implies that $\mathrm{S_pcl}(\{x\}) \cap \mathrm{S_pcl}(\{y\})$.

Conversely: Let the hypothesis be satisfied and let U be any $\mathrm{S_p}$-open set in X and $x \in U$. If $U = X$, then clearly $\mathrm{S_pcl}(\{x\}) \subseteq U$, but if $U \neq X$, then there exists $y \in X$ such that $y \notin U$. Now $x \neq y$ and $x \notin \mathrm{S_pcl}(\{y\})$ implies that $\mathrm{S_pcl}(\{x\}) \neq \mathrm{S_pcl}(\{y\})$, then by hypothesis $\mathrm{S_pcl}(\{x\}) \cap \mathrm{S_pcl}(\{y\}) = \phi$ implies that $y \notin \mathrm{S_pcl}(\{x\})$. Thus if $y \notin U$, then $y \notin \mathrm{S_pcl}(\{x\})$ this implies that $\mathrm{S_pcl}(\{x\}) \subseteq U$. Hence X is $\mathrm{S_p}$-$\mathrm{R_0}$ space.

Theorem 3.14: A space X is $\mathrm{S_p}$-$\mathrm{R_0}$ if and only if for any points $x, y \in X$, if $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \neq \mathrm{S_p}$-$\mathrm{ker}(\{y\})$, then $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \cap \mathrm{S_p}$-$\mathrm{ker}(\{y\}) = \phi$.

Proof: Let X be $\mathrm{S_p}$-$\mathrm{R_0}$ space and $x, y \in X$ such that $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \neq \mathrm{S_p}$-$\mathrm{ker}(\{y\})$. Then by [theorem 3.5] $\mathrm{S_pcl}(\{x\}) \neq \mathrm{S_pcl}(\{y\})$, now to show $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \cap \mathrm{S_p}$-$\mathrm{ker}(\{y\}) = \phi$, if possible suppose that $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \cap \mathrm{S_p}$-$\mathrm{ker}(\{y\}) \neq \phi$ implies that there exist $z \in \mathrm{S_p}$-$\mathrm{ker}(\{x\}) \cap \mathrm{S_p}$-$\mathrm{ker}(\{y\})$, then $z \in \mathrm{S_p}$-$\mathrm{ker}(\{x\})$ and $z \in \mathrm{S_p}$-$\mathrm{ker}(\{y\})$ implies that by [Lemma 3.4] $x \in \mathrm{S_pcl}(\{z\})$ and since $x \in \mathrm{S_pcl}(\{x\})$, then by [Theorem 3.13] $\mathrm{S_pcl}(\{z\}) = \mathrm{S_pcl}(\{x\})$ similarly $\mathrm{S_pcl}(\{z\}) = \mathrm{S_pcl}(\{y\})$ this implies that $\mathrm{S_pcl}(\{x\}) = \mathrm{S_pcl}(\{y\})$ is a contradiction, thus $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \cap \mathrm{S_p}$-$\mathrm{ker}(\{y\}) = \phi$.

Conversely: Let the hypothesis be satisfied. To show X is $\mathrm{S_p}$-$\mathrm{R_0}$ space, let $x, y \in X$ such that $\mathrm{S_pcl}(\{x\}) \neq \mathrm{S_pcl}(\{y\})$, then by [Theorem 3.5] $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \neq \mathrm{S_p}$-$\mathrm{ker}(\{y\})$ implies that by hypothesis $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \cap \mathrm{S_p}$-$\mathrm{ker}(\{y\}) = \phi$ which is implies that $\mathrm{S_pcl}(\{x\}) \cap \mathrm{S_pcl}(\{y\}) = \phi$. Thus by [Theorem 3.13] X is $\mathrm{S_p}$-$\mathrm{R_0}$ space.

Theorem 3.15: For a space X, the following are equivalent:

1. X is $\mathrm{S_p}$-$\mathrm{R_0}$ space.
2. For any $A \neq \phi$ and $G \in \mathrm{S_pO}(X)$ such that $G \cap A \neq \phi$, there exists an $\mathrm{S_p}$-closed set F such that $A \cap F \neq \phi$ and $F \subseteq G$.
3. Any $\mathrm{S_p}$-open set G, $G = \cup\{F : F \in \mathrm{S_pC}(X) \text{ and } F \subseteq G\}$.
4. Any $\mathrm{S_p}$-closed set F, $F = \cap\{G \in \mathrm{S_pO}(X) : F \subseteq G\}$.
5. For any x, $\mathrm{S_pcl}(\{x\}) \subseteq \mathrm{S_p}$-$\mathrm{ker}(\{x\})$.

Proof: (1) ⇒ (2)**.** Let A be a non-empty set in X and $G \in \mathrm{S_pO}(X)$ such that $A \cap G \neq \phi$. Then there exist $x \in A \cap G$ implies that $x \in G$ and since X is $\mathrm{S_p}$-$\mathrm{R_0}$ so $\mathrm{S_pcl}(\{x\}) \subseteq G$. Let $F = \mathrm{S_pcl}(\{x\})$ which is $\mathrm{S_p}$-closed set such that $F \cap G \neq \phi$ and $F \subseteq G$.

(2) ⇒ (3). Let G be any $\mathrm{S_p}$-open set in X. Then clearly $\cup\{F \in \mathrm{S_pC}(X) : F \subseteq G\} \subseteq G$, and let $x \in G$. Then take $A = \{x\}$ and $A \cap G \neq \phi$ so by hypothesis there exists an $\mathrm{S_p}$-closed set F such that $F \subseteq G$ and $x \in F$. Now $x \in F \subseteq \cup\{F \in \mathrm{S_pC}(X) : F \subseteq G\}$ this implies that $G \subseteq \cup\{F \in \mathrm{S_pC}(X) : F \subseteq G\}$. Thus $G = \cup\{F : F \in \mathrm{S_pC}(X) \text{ and } F \subseteq G\}$.

(3) ⇒ (4). Obvious.

(4) ⇒ (5). Let x be any point in the space X and $y \notin \mathrm{S_p}$-$\mathrm{ker}(\{x\})$. Then there exists an $\mathrm{S_p}$-open set U such that $x \in U$ and $y \notin U$ implies that $\mathrm{S_pcl}(\{y\}) \cap U = \phi$ and then by hypothesis $\cap\{G \in \mathrm{S_pO}(X) : \mathrm{S_pcl}(\{y\}) \subseteq G\} \cap U = \phi$ this implies that there exists an $\mathrm{S_p}$-open set G such that $x \notin G$ and $\mathrm{S_pcl}(\{y\}) \subseteq G$. Therefore $\mathrm{S_pcl}(\{x\}) \cap G = \phi$ and $y \notin \mathrm{S_pcl}(\{x\})$, consequently we obtain $\mathrm{S_pcl}(\{x\}) \subseteq \mathrm{S_p}$-$\mathrm{ker}(\{x\})$.

(5) ⇒ (1). Let G be any $\mathrm{S_p}$-open set in X and $x \in G$ and let $y \in \mathrm{S_p}$-$\mathrm{ker}(\{x\})$. Then by [Lemma 3.4] $x \in \mathrm{S_pcl}(\{x\})$ and $y \in G$. But clearly $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \subseteq G$, then by hypothesis $\mathrm{S_pcl}(\{x\}) \subseteq \mathrm{S_p}$-$\mathrm{ker}(\{x\}) \subseteq G$ this implies that $\mathrm{S_pcl}(\{x\}) \subseteq G$. Hence X is $\mathrm{S_p}$-$\mathrm{R_0}$ space.

Corollary 3.16: A space X is $\mathrm{S_p}$-$\mathrm{R_0}$ if and only if for all $x \in X$ $\mathrm{S_pcl}(\{x\}) = \mathrm{S_p}$-$\mathrm{ker}(\{x\})$.

Proof: Let X be $\mathrm{S_p}$-$\mathrm{R_0}$. Then by [Theorem 3.15] for any $x \in X$, $\mathrm{S_pcl}(\{x\}) \subseteq \mathrm{S_p}$-$\mathrm{ker}(\{x\})$, and now let $y \in \mathrm{S_p}$-$\mathrm{ker}(\{x\})$ implies that by [Lemma 3.4] $x \in \mathrm{S_pcl}(\{y\})$, then $\mathrm{S_pcl}(\{x\}) \cap \mathrm{S_pcl}(\{y\}) \neq \phi$ implies that by [Theorem 3.13] $\mathrm{S_pcl}(\{x\}) = \mathrm{S_pcl}(\{y\})$ and then $y \in \mathrm{S_pcl}(\{x\})$, thus $\mathrm{S_pcl}(\{x\}) = \mathrm{S_p}$-$\mathrm{ker}(\{x\})$.

Conversly: Obvious from [Theorem 3.15].

Theorem 3.17: For a space X, the following statements are equivalent:

1. X is $\mathrm{S_p}$-$\mathrm{R_0}$ space.
2. For any $\mathrm{S_p}$-closed set F, then $F = \mathrm{S_p}$-$\mathrm{ker}(F)$.
3. If F is $\mathrm{S_p}$-closed set and $x \in F$, then $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \subseteq F$.

Proof: (1) ⇒ (2). Let F be any $\mathrm{S_p}$-closed set in X and $x \notin F$. Then $X \backslash F$ is $\mathrm{S_p}$-open set contain x and since X is $\mathrm{S_p}$-$\mathrm{R_0}$ space so $\mathrm{S_pcl}(\{x\}) \subseteq X \backslash F$ implies that $\mathrm{S_pcl}(\{x\}) \cap F = \phi$, then by [Lemma 3.3] $x \notin \mathrm{S_p}$-$\mathrm{ker}(F)$, therefore $F = \mathrm{S_p}$-$\mathrm{ker}(F)$.

(2) ⇒ (3). Ingeneral if $A \subseteq B$, then $\mathrm{S_p}$-$\mathrm{ker}(A) \subseteq \mathrm{S_p}$-$\mathrm{ker}(B)$ so from (2) if $x \in F$, where F is $\mathrm{S_p}$-closed in X, then $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \subseteq \mathrm{S_p}$-$\mathrm{ker}(F) = F$ implies that $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \subseteq F$.

(3) ⇒ (1). Let $x, y \in X$ and $x \in \mathrm{S_pcl}(\{y\})$, then by [Lemma 3.4] $y \in \mathrm{S_p}$-$\mathrm{ker}(\{x\})$ and since $x \in \mathrm{S_pcl}(\{x\})$, but $\mathrm{S_pcl}(\{x\})$ is $\mathrm{S_p}$-closed set so by (3) $\mathrm{S_p}$-$\mathrm{ker}(\{x\}) \subseteq \mathrm{S_pcl}(\{x\})$ this implies that $y \in \mathrm{S_pcl}(\{x\})$ and for the converse it is obvious. Thus by [Theo-

rem 3.12] X is S_p-R_0 space.

Corollary 3.18: A space X is S_p-R_1 if and only if for any two points $x, y \in X$ such that S_p-ker$(\{x\}) \neq S_p$-ker$(\{y\})$, there exist disjoint S_p-open sets U and V such that $S_p\text{cl}(\{x\}) \subseteq U$ and $S_p\text{cl}(\{y\}) \subseteq V$.

Proof: Let X be S_p-R_1 space and $x, y \in X$ such that S_p-ker$(\{x\}) \neq S_p$-ker$(\{y\})$. Then by [Theoren 3.5] $S_p\text{cl}(\{x\}) \neq S_p\text{cl}(\{y\})$, then there exist two disjoint S_p-open set U and V such that $S_p\text{cl}(\{x\}) \subseteq U$ and $S_p\text{cl}(\{y\}) \subseteq V$.

Conversely: Let $x, y \in X$ such that $S_p\text{cl}(\{x\}) \neq S_p\text{cl}(\{y\})$. Then by [Theoren 3.5] S_p-ker$(\{x\}) \neq S_p$-ker$(\{y\})$ implies that by hypothesis there exist two disjoint S_p-open sets U and V such that $S_p\text{cl}(\{x\}) \subseteq U$ and $S_p\text{cl}(\{y\}) \subseteq V$. Thus X is S_p-R_1 space.

Theorem 3.19: A space X is S_p-T_1 if and only if it is S_p-T_0 and S_p-R_0.

Proof: Let X be S_p-T_1 space. Then from [Definition 2.8] and [Definition 2.7] X is S_p-T_0 and by [Theorem 2.10] every singlten set in X is S_p-closed. Now X is S_p-R_0 space since for any $x \in G$, where G is S_p-open set, $S_p\text{cl}(\{x\}) = \{x\} \subseteq G$. Thus the space X is S_p-T_0 and S_p-R_0.

Conversley: Let $x, y \in X$ be any two distinct points. Since X is S_p-T_0 space so there exists an S_p-open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$. Now let $x \in U$ and $y \notin U$ and since X is S_p-R_0 space so $S_p\text{cl}(\{x\}) \subseteq U$ and we have $y \notin U$ implies that $y \notin S_p\text{cl}(\{x\})$, then $y \in X \setminus S_p\text{cl}(\{x\})$ which is S_p-open set so take $V = X \setminus S_p\text{cl}(\{x\})$. Thus U and V are S_p-open sets in X such that $x \in U$, $y \in V$ and $x \notin V$ and $y \notin U$, implies that X is S_p-T_1 space.

Theorem 3.20: A space X is S_p-T_2 if and only if it is S_p-R_1 and S_p-T_0.

Proof: Let X be S_p-T_2. Then from Definition 2.9 and Definition 2.7 X is S_p-T_0 and to show X is S_p-R_1 space let $x, y \in X$ such that $S_p\text{cl}(\{x\}) \neq S_p\text{cl}(\{y\})$ and since X is S_p-T_1 space so by [Theorem 2.10] every singleton set in X is S_p-closed, that is meaning $S_p\text{cl}(\{x\}) = \{x\}$ and $S_p\text{cl}(\{y\}) = \{y\}$ implies that $\{x\} \neq \{y\}$ and since X is S_p-T_2 space so there exist two disjoint S_p-open sets U and V such that $x \in U$ and $y \in V$ implies that $S_p\text{cl}(\{x\}) \subseteq U$ and $S_p\text{cl}(\{y\}) \subseteq V$. Thus X is S_p-R_1 space.

Conversely: Let X be S_p-R_1 and S_p-T_0 space and $x, y \in X$ such that $x \neq y$. Now since X is S_p-T_0 so by [Definition 2.7] there exist an S_p-open set U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$, take $x \in U$ and $y \notin U$ implies that $U \cap \{y\} = \phi$, and then $x \notin S_p\text{cl}(\{y\})$ this implies that $S_p\text{cl}(\{x\}) \neq S_p\text{cl}(\{y\})$ and since X is S_p-R_1 so there exist two disjoint S_p-open sets U and V such that $S_p\text{cl}(\{x\}) \subseteq U$ and $S_p\text{cl}(\{y\}) \subseteq V$ implies that $x \in U$ and $y \in V$. Thus X is S_p-T_2 space.

Definition 3.21: [1] A filterbase F is said to be S_p-converges to a point $x \in X$, if for each S_p-open set U containing x, there exists $B \in F$ such that $B \subseteq U$.

Lemma 3.22: Let X be a space and let x and y be any two points in X such that every net in X S_p-converging to y S_p-converging to x, then $x \in S_p\text{cl}(\{y\})$.

Proof: Let $x_n = y$ for all $n \in N$. Then $\{x_n\}_{n \in N}$ is a net in $S_p\text{cl}(\{y\})$, clearly $\{x_n\}_{n \in N}$ is S_p-converges to y and then by the hypothesis $\{x_n\}_{n \in N}$ is S_p-converges to x this implies that $x \in S_p\text{cl}(\{y\})$.

Theorem 3.23: For a space X, the following statements are equivalent:

1. X is S_p-R_0 space.
2. If $x, y \in X$, then $y \in S_p\text{cl}(\{x\})$ if and only if every net in X S_p-converging to y S_p-converging to x.

Proof: (1) ⇒ (2). Let $x, y \in X$ such that $y \in S_p\text{cl}(\{x\})$ and let $\{x_\alpha\}_{\alpha \in \Delta}$ be a net in X such that S_p-converges to y. Since $y \in S_p\text{cl}(\{x\})$, then by [Theorem 3.13] $S_p\text{cl}(\{x\}) = S_p\text{cl}(\{y\})$ implies that $x \in S_p\text{cl}(\{y\})$ and then $\{x_\alpha\}_{\alpha \in \Delta}$ is S_p-converges to x.

Conversely: Let $x, y \in X$ such that every net in X S_p-converging to y S_p-converging to x. Then by [Lemma 3.22] $x \in S_p\text{cl}(\{y\})$ and by [Theorem 3.13] we have $S_p\text{cl}(\{x\}) = S_p\text{cl}(\{y\})$ this implies that $y \in S_p\text{cl}(\{x\})$.

(2) ⇒ (1). Let $x, y \in X$ such that $S_p\text{cl}(\{x\}) \cap S_p\text{cl}(\{y\}) \neq \phi$. Then there exist $w \in S_p\text{cl}(\{x\}) \cap S_p\text{cl}(\{y\})$ implies that there exist a net $\{x_\alpha\}_{\alpha \in \Delta}$ in $S_p\text{cl}(\{x\})$ such that $\{x_\alpha\}_{\alpha \in \Delta}$ is S_p-converges to w and since $w \in S_p\text{cl}(\{y\})$, then by hypothesis $\{x_\alpha\}_{\alpha \in \Delta}$ is S_p-converges to y this implies that $y \in S_p\text{cl}(\{x\})$. By the same token we obtain that $x \in S_p\text{cl}(\{y\})$ implies that $S_p\text{cl}(\{x\}) = S_p\text{cl}(\{y\})$ and then by [Theorem 3.13] X is S_p-R_0 space.

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